JGP - Vol. 2, n. 1, 1985

Invariant star-products on symplectic manifolds

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Abstract. *Let (M, F) be a symplectic manifold and consider a Lie subalgebra $\mathbb{G}$ of its Lie algebra of symplectic vector fields. We prove that every one-differentiable deformation of order k of the Poisson Lie algebra of M, which is invariant with respect to $\mathbb{G}$, extends to an invariant one-differentiable deformation of infinite order. If M admits a $\mathbb{G}$-invariant linear connection, a similar result holds true for differentiable deformations and for star-products. In particular, if M admits a $\mathbb{G}$-invariant linear connection, there always exists a $\mathbb{G}$-invariant star-product.*

INTRODUCTION

Let M be a smooth connected Hausdorff second countable manifold equipped with a symplectic form F. We assume that $\dim M > 2$. Denote by $\mathbf{L}$ the Lie algebra of symplectic vector fields of (M, F).

Let $\mathbb{G}$ be a Lie subalgebra of $\mathbf{L}$. The aim of this paper is to study the formal deformations of the Poisson Lie algebra (N, P) where N is the space of all smooth real functions on M and P the Poisson bracket, and the star-products which are invariant by $\mathbb{G}$.

This problems has already been considered by various authors, namely [6, 5, 1]. It is shown in [6] that, if there exists an invariant Vey star-product, then M admits an invariant symplectic connection. We prove here that the existence of an invariant linear connection implies that every invariant formal deformation of P or star-product of finite order extends to an invariant formal deformation of

Key Words: Symplectic manifold, deformations of the Poisson Lie algebra, star-products, invariance.

1980 Mathematics Subject Classification: 53 C 15.

P or star-product (of infinite order).

It is proved in [6], § 18, that, for the Hochschild cohomology of N, an invariant coboundary is the coboundary of an invariant cochain. However, using then an argument of the Neroslavsky-Vlassov type [8] leads to obstructions lying in the third invariant de Rham cohomology space of M. In this note, we avoid these obstructions essentially by adapting to the invariant case the proof of [2] of the existence of star-products on an arbitrary symplectic manifold. To do this, we need the description of the invariant Chevalley cohomology of (N, P) obtained in [3]. The adaptation is straightforward in the case of 1-differentiable deformations but more delicate in the general case.

We use the definitions and notations of [2].

THE MAP τ

Recall that $A^p_{\text{diff},nc}(N)$ is the space of differential cochains on N (i.e. alternating $(p+1)$-linear maps of N^{p+1} into N), vanishing on the constants (nc); $A^p_{\text{diff}}(\mathcal{X}(M), N)$ is the space of differential N-valued $(p+1)$-cochains on the Lie algebra of all vector fields $\mathcal{X}(M)$.

Then $\mu^* : A^p_{\text{diff}}(\mathcal{X}(M), N) \to A^p_{\text{diff},nc}(N)$ is defined by

$$\mu^* c(u_0, \ldots, u_p) = c(H_{u_0}, \ldots, H_{u_p})$$

where H_u is the Hamiltonian vector field of u.

One of the keys in [2] is the existence of a right inverse τ of μ^* for $p = 1$, with appropriate additional properties. One more is required here: that τ preserves invariance.

Denote by ∂ and ∂' the Chevalley coboundary operators of $A_{\text{diff}}(N)$ and $A_{\text{diff}}(\mathcal{X}(M), N)$ and by S^3_Γ and ϕ_Γ the Vey cocycle of $A^1_{\text{diff},nc}(N)$ and the cocycle of $A^1_{\text{diff}}(\mathcal{X}(M), \Lambda^2(M))$ such that $S^3_\Gamma = \mu^* \langle \Lambda, \phi_\Gamma \rangle$, where Λ is the contravariant analogue of F.

PROPOSITION 1. *Let M admit a $\mathbb{G}$-invariant connection Γ. Then there exists a linear map $\tau : A^1_{\text{diff},nc}(N) \to A^1_{\text{diff}}(\mathcal{X}(M), N)$ such that*

(i) $\mu^* \circ \tau = \text{id}$ *on* $A^1_{\text{diff},nc}(N)$,

(ii) $\tau \circ \mu^* = \text{id}$ *on* $\Lambda^2(M)$

(iii) $\tau(f S^3_\Gamma) = f \langle \Lambda, \phi_\Gamma \rangle, \quad \forall f \in N$,

(iv) $(\tau \circ \partial)\, A^0_{\text{diff},nc}(N) \subset \text{im}\, \partial'$

(v) *if* $C \in A^1_{\text{diff},nc}(N)$ *is* $\mathbb{G}$*-invariant,* τC *is* $\mathbb{G}$*-invariant.*

It is known [6] that each $C \in A^1_{\mathrm{diff},nc}(N)$ can be written (in a unique way)

$$C(u,v) = \sum_{p,q} \langle T^{p,q}, \nabla^p u \otimes \nabla^q v \rangle$$

where $T^{p,q}$ is a contravariant $(p+q)$-tensor, symmetric in its first p and its last q arguments and where $\langle\, , \rangle$ indicates the contraction of the first p indices of $T^{p,q}$ with $\nabla^p u$ and of the last one's with $\nabla^q v$; since ∇, the covariant derivative corresponding to Γ, is invariant, C is invariant if and only if each $T^{p,q}$ is invariant. Thus $\tau' C$, obtained from C by replacing

$$\nabla^p u = \nabla^{p-1} \mathrm{d}u = \nabla^{p-1} i(H_u) F$$

by

$$X \to \nabla^{p-1} i(X) F$$

transforms an invariant element of $A^1_{\mathrm{diff},nc}(N)$ into an invariant element of $A^1_{\mathrm{diff},nc}(N)$, and τ' is a right inverse of μ^*. Unfortunately, it does not verify (iii) and (iv). Clearly, τ' can be defined similarly on each $A^p_{\mathrm{diff},nc}(N)$.

Denote by I the subspace of all invariant elements of $A^1_{\mathrm{diff},nc}(N)$ and set

$$A^1_{\mathrm{diff},nc}(N) = [\partial A^0_{\mathrm{diff},nc}(N) + I + \mu^* \Lambda^2(M) + N \cdot S^3_\Gamma] \oplus E_1 .$$

We can write

$$\partial A^0_{\mathrm{diff},nc}(N) + I + \mu^* \Lambda^2(M) + N \cdot S^3_\Gamma = \mu^* \Lambda^2(M) \oplus N \cdot S^3_\Gamma \oplus E_2 \oplus E_3 \oplus E_4$$

where $E_2 \subset I \cap \mathrm{im}\, \partial$, $E_3 \subset \mathrm{im}\, \partial \setminus I$, $E_4 \subset I \setminus \mathrm{im}\, \partial$.

We now define τ as follows:

(i) on E_1, τ is any right inverse of μ^*,
(ii) on E_4, $\tau = \tau'$,
(iii) on E_3, τ is defined as in [2],
(iv) on $\mu^* \Lambda^2(M)$, $\tau \circ \mu^* \Omega = \Omega$,
(v) on NS^3_Γ, $\tau(f S^3_\Gamma) = f \langle \Lambda, \phi_\Gamma \rangle$,
(vi) on E_2: according to [3], prop. 7.1., if $C = \partial D \in A^1_{\mathrm{diff},nc}(N)$ is invariant, $C = \partial(E + \mu^* \omega)$, where E is invariant and $\omega \in \Lambda^1(M)$. There exist thus $k_1 : E_2 \to A^0_{\mathrm{diff},nc}(N)$ and $k_2 : E_2 \to \Lambda^1(M)$ such that $k_1 E_2$ is made of invariant elements and $\partial \circ (k_1 + \mu^* \circ k_2) = \mathrm{id}$ on E_2. Set $\tau = \partial' \circ \tau' \circ k_1 + \partial' \circ k_2$.

Then τ verifies all the required properties.

THE ONE-DIFFERENTIABLE CASE

Recall that a *formal differential deformation of order* k *of* P is a formal series

$\mathcal{L}_\nu = \sum_{i=0}^{\infty} \nu^i C_i$ of differential bilinear maps $C_i : N \times N \to N$, vanishing on the constants and such that the formal Jacobi identity is verified up to the order k. In other words, [,] denoting the Nijenhuis-Richardson bracket, the components of ν^i ($i \leqslant k$) in $[\mathcal{L}_\nu, \mathcal{L}_\nu]$ are vanishing.

The first result in [2] essentially shows the following. If $\mathcal{L}_\nu$ is a differential deformation of P and if Ω is closed, the differential deformation of order k $\mathcal{L}_\nu + \nu^k \mu^* \Omega$ extends to a differential deformation. The same holds true within the class of invariant deformations. The results of [2] are stated for local deformations but apply trivially to differential deformations.

THEOREM 2. *Assume that M admits a $\mathbb{G}$-invariant linear connection. Let $\mathcal{L}_\nu$ be a differential $\mathbb{G}$-invariant deformation of P and let Ω be a closed $\mathbb{G}$-invariant 2-form. Define inductively $\mathbb{L}_\nu^k$ ($k \in \mathbb{N}$) by $\mathbb{L}_\nu^0 = \mathcal{L}_\nu$ and*

$$\ell \, \mathbb{L}_\nu^\ell |_U = \sum_{p+q=\ell-1} [\mathbb{L}_\nu^p, \mu^* i(X) \, \tau \, \mathbb{L}_\nu^q] |_U$$

whenever $\Omega = \mathrm{d}i(X)F$ on the open subset U of M. Then each $\mathbb{L}_\nu^k$ is globally defined, $\mathbb{G}$-invariant, and

$$\sum_{k=0}^{\infty} \mu^k \mathbb{L}_\nu^k$$

is a differential deformation in μ of $\mathcal{L}_\nu$. In particular,

$$\sum_{k=0}^{\infty} \nu^{kt} \mathbb{L}_\nu^k$$

is a $\mathbb{G}$-invariant deformation of P equal to $\mathcal{L}_\nu + \nu^t \mu^* \Omega$ at the order t.

The only point added to thm. 2.1. in [2] is the $\mathbb{G}$-invariance. We prove it by induction on ℓ. Since Ω is invariant, for $Y \in \mathbb{G}$, on the open subset U,

$$0 = L_Y \Omega = L_Y \mathrm{d}i\,(X)\, F = \mathrm{d}i\,(L_Y X)\, F = L_{L_Y X} F.$$

Thus $L_Y X \in \mathrm{L}$. Assume now that $\mathbb{L}_\nu^p$ is $\mathbb{G}$-invariant for $p < \ell$. Then, since $\mathbb{L}_\nu^q$ and $\tau \mathbb{L}_\nu^q$ are invariant, if U is contractible and $L_Y X = H_u$ on U,

$$\ell \, L_Y \mathbb{L}_\nu^\ell |_U = \sum_{p+q=\ell-1} [\mathbb{L}_\nu^p, \mu^* i(L_Y X) \, \tau \, \mathbb{L}_\nu^q] =$$

$$= \frac{1}{2} i(u) \sum_{p+q=\ell-1} [\mathbb{L}_\nu^p, \mathbb{L}_\nu^q] = 0$$

since $\Sigma\, \mu^k \mathbb{L}_\nu^k$ is a formal deformation of $\mathcal{L}_\nu$.

Remark 3. As mentioned in [2], if $\mathcal{L}_\nu$ is 1-differentiable, so are all the $\mathbb{L}_\nu^k$'s. In this case, we make use of τ only on $\mu^*\Lambda^2(M)$, where it is uniquely defined by $\mu^* \circ \tau = \mathrm{id}$. Thus *thm. 2. holds true for 1-differentiable deformations without the assumption that M admits a* $\mathbb{G}$*-invariant connection.* As in [2], it follows that *every* $\mathbb{G}$*-invariant 1-differentiable deformation of order k extends to a* $\mathbb{G}$*-invariant 1-differentiable deformation.*

INVARIANT DIFFERENTIAL DEFORMATIONS WITH DRIVER $P + \nu r S_\Gamma^3$

The coefficient of ν in a differential deformation of P is a cocycle of $A^1_{\mathrm{loc},nc}(N)$, thus it has the form $C = r\, S_\Gamma^3 + \mu^*\Omega + \partial E$. A basic step in contructing a differential deformation equal to $P + \nu C$ at the order 1 is to obtain a differential deformation equal to $P + \nu r S_\Gamma^3$ at the order 1. Its existence is granted by thm. 3.3. of [2] and what we are going to prove now is that, if Γ is invariant, the differential deformation constructed in [2] is also invariant.

THEOREM 4. *Assume that M admits a* $\mathbb{G}$*-invariant linear connection. Then, given* $r \in \mathbb{R}_0$*, there exists a unique differential deformation* $\mathcal{L}_\nu$ *of P with driver* $P +$ $+ \nu r S_\Gamma^3$*, such that*

$$(1) \qquad \partial_\nu \theta + \mathbb{L}_\nu^1(\mathcal{L}_\nu, F) = 0,$$

where $\mathbb{L}_\nu^1(\mathcal{L}_\nu, F)$ *is defined as in thm. 2 for* $\Omega = F$ *and* θ *is the map*

$$\Sigma\, \nu^k u_k \to \Sigma\, (2k-1)\, \nu^k u_k\,.$$

Moreover, this $\mathcal{L}_\nu$ *is* $\mathbb{G}$*-invariant.*

The result is proved in [2], thm. 3.3, except the invariance of $\mathcal{L}_\nu = \Sigma\, \nu^k C_k$. We will prove by induction on k that C_k is invariant. It is true for $k = 0,1$. As in [2], ξ is defined on a contractible open subset U by $F|_U = \mathrm{di}\,(\xi)\, F$.

Recall that a 2-cocycle C has a unique decomposition $C = r' S_\Gamma^3 + \mu^* D$, where D is a cocycle; thus $C = \mu^* \hat{C}$, where $\hat{C} = r' \langle \Lambda, \phi_\Gamma \rangle + D$. The cocycle D is not unique but, whatever it is, $\mu^* i\,(\xi)\, \partial' \hat{C} = -r' S_\Gamma^3$. Thus $pC + \mu^* i\,(\xi)\, \partial' \hat{C} = 0$ implies $C = 0$ provided $p \neq 0$ and 1.

Consider now C_k. The relation (1) implies that

$$L_\xi C_k + (2k+1) C_k - \partial \mu^* i\,(\xi)\, \tau C_k = \sum_{\substack{p+q=k \\ p,q>0}} [C_p, \mu^* i\,(\xi)\, \tau C_q]$$

and standard computations (see [2], lemma 3.1) reduce this equality to

$$(2k-1)C_k + \mu^* i(\xi)\partial' \tau C_k = \sum_{\substack{p+q=k \\ p,q>0}} [C_p, \mu^* i(\xi)\tau C_q].$$

Apply $L_X(X \in \mathbb{G})$ to both sides. If $X = H_u$ on U, $L_X \xi = H_{u - L_\xi u}$. We thus obtain, assuming that C_p is invariant for $p < k$,

$$(2k-1)L_X C_k + \mu^* i(\xi)\partial L_X \tau C_k = -\frac{1}{2} i(u - L_\xi u)[\mathcal{L}_\nu, \mathcal{L}_\nu]_k = 0. \tag{2}$$

On the other hand, ∂C_k is invariant. Thus, by [3], prop. 8.1.

$$C_k = f S^3_\Gamma + \mu^*\Omega + E + \partial E',$$

where E, $\mathrm{d}f$ and $\mathrm{d}\Omega$ are invariant. Then

$$\tau C_k = f \langle \Lambda, \phi_\Gamma \rangle + \Omega + \tau E + \tau \partial E'$$

where τE is invariant and $\tau \partial E'$ is a coboundary and

$$\mu^* L_X \tau C_k = L_X f \cdot S^3_\Gamma + \mu^*(L_X \Omega + L_X \tau \partial E'),$$

where $L_X f$ is constant and $L_X(\Omega + \tau \partial E')$ is a cocycle.

Since $2k - 1 > 1$, (2) implies that $L_X C_k = 0$. Hence the result.

THE GENERAL CASE

THEOREM 5. *Let M admit a $\mathbb{G}$-invariant linear connection. Then every invariant formal deformation of P of any order is the driver of an invariant deformation of P.*

We refer to the proof of thm. 3.4 of [2] and observe that every invariant 2-cocycle can be written $rS^3_\Gamma + \mu^*\eta + \partial T$, where η and T are invariant. The proof is then entirely similar to the above-mentioned one, using thm. 2 and 4.

STAR-PRODUCTS

A differential *weak star-product (resp. star-product)* of order k is a series $M_\lambda = \Sigma \lambda^i C_i$ of differential bilinear maps $C_i : N \times N \to N$ symmetric (resp. and *nc*) for i even $\geqslant 2$, antisymmetric and *nc* for i odd, such that $C_0 = \mathbf{m}: (u, v) \to uv$, $C_1 = P$ and such that M_λ is associative up to the order k.

It is shown in [6], §5 that if M_λ is a weak star-product of order k, its terms

$C_{2i}(2i < k)$ take the form $C_{2i} = \bar{C}_{2i} + a_i \mathrm{m}$ with $a_i \in \mathbb{R}$ and $\bar{C}_i$ *nc*. Denote by $\mathbb{P}_\lambda$ the set of all formal series $1 + \sum_{i=1}^{\infty} a_i \lambda^i$ ($a_i \in \mathbb{R}$). It is then easily seen that each weak star-product M_λ can be factorized in a unique way

(3) $$M_\lambda = p(\lambda^2)\, \bar{M}_\lambda$$

where $p(\lambda) \in \mathbb{P}_\lambda$ and $\bar{M}_\lambda$ is a star-product.

From each weak star-product (resp. of order $2k$) M_λ derives a formal deformation of P (resp. of order $k-1$)

$$\mathcal{L}_\nu^M(u, v) = \frac{1}{2\nu}\, [M_\lambda(u, v) - M_\lambda(v, u)]_{\lambda = \nu^2}.$$

Conversely, a formal deformation $\mathcal{L}_\nu = \Sigma\, \nu^i C_{2i+1}$ derives from a (unique) week star-product if and only if

$$C_3 = \frac{1}{3!}\, S_\Gamma^3 + \mu^* \Omega + \partial E,$$

where $d\Omega = 0$ ([4], thm. 3.5).

It follows from (3) that, if $\mathcal{L}_\nu$ *derives from* M_λ, M_λ *is a star product if and only if* $\mathcal{L}_\nu$ *has no divisor in* $\mathbb{P}_\nu \setminus \{1\}$.

The existence of invariant star-products is now ruled by the following theorem. The case of weak star-products is easily deduced by (3).

THEOREM 6. *Assume that M admits a $\mathbb{G}$-invariant linear connection Γ. Then every invariant differential star-product of order $2k$ is the driver of order $2k$ of an invariant star-product. In particular, M admits at least one invariant star--product.*

Let $M_\lambda^k = \sum_{i=0}^{2k} \lambda^i C_i$ be an invariant differential star-product of order $2k$.

Assume first that $k > 1$. If $\mathcal{L}_\nu^{k-1}$ derived from M_λ^k, it is invariant and thus it extends to an invariant $\mathcal{L}_\nu$. This $\mathcal{L}_\nu$ derives from a weak star-product $M_\lambda = \sum_{i=0}^{\infty} \lambda^i C_i'$. It is easily seen that M_λ is differential. It is invariant. Indeed, with the notations of [4], if $X \in \mathbb{G}$

$$0 = L_X(M_\lambda \Delta M_\lambda) = 2\, M_\lambda \Delta L_X M_\lambda$$

and $L_X M_\lambda = \sum_{i=1}^{\infty} \lambda^{2i} L_X C_{2i}'$. Thus, by [4], cor. 3.7, $L_X M_\lambda = 0$.

We have now to check if M_λ extends M_λ^k.

If $i < k-1$, $C_{2i}' = C_{2i}$. This is proved by induction on i as follows. If it is true for $j < k - 2$, we have

$$P\,\Delta\,(C_{2j+2} - C'_{2j+2}) = 0,$$

$$\mathrm{m}\,\Delta\,(C_{2j+4} - C'_{2j+4}) + C_2\,\Delta\,(C_{2j+2} - C'_{2j+2}) = 0,$$

$$P\,\Delta\,(C_{2j+4} - C'_{2j+4}) + C_3\,\Delta\,(C_{2j+2} - C'_{2j+2}) = 0$$

hence, by [4], lemma 3.6, $C_{2j+2} = C'_{2j+2}$.

For $i = k - 1$, we only have

$$P\,\Delta\,(C_{2k-2} - C'_{2k-2}) = 0$$

thus, by the same lemma,

$$C_{2k-2} = C'_{2k-2} + a\,\mathrm{m}\ (a \in \mathbb{R}).$$

Replacing M_λ by $(1 - a\lambda^{2k-2})M_\lambda$, we obtain $C'_i = C_i$ for $i \leqslant 2k - 2$, but now $C'_{2k-1} = C_{2k-1} - a\,P$.

Define

$$\pi : \Sigma\ \lambda^k u_k \to \Sigma\ k\,\lambda^k u_k.$$

Then $\mathrm{Ad}\,(\exp \lambda^t b\pi)M_\lambda$ is a new weak star-product equal to M_λ up to the order t and its term of order $t + 1$ is $C'_{t+1} + bP$. Thus replacing again M_λ by $\mathrm{Ad}\,(\exp \lambda^{2k} a\pi)M_\lambda$, we obtain a new M_λ equal to M_λ^k up to the order $2k - 1$ and still invariant.

For the term of order $2k$, we have now

$$\mathrm{m}\,\Delta\,(C'_{2k} - C_{2k}) = 0$$

hence $C'_{2k} - C_{2k}$ is a Hochschild 2-cocycle. Being symmetric, it is a coboundary. It is moreover invariant. By [6], it is of the type $\mathrm{m}\Delta T$ for some invariant T. Transforming now M_λ by $\mathrm{Ad}\,(1 + \lambda^{2k} T)$, we finally obtain a weak star-product M_λ equal to M_λ^k up to the order $2k$ and invariant. The related starproduct $\overline{M}_\lambda$ is still invariant and equal to M_λ^{2k} up to the order $2k$, hence the result.

If $k = 1$, according to [6], $M_\lambda^1 = \mathrm{m} + \lambda P + \dfrac{1}{2}\,\lambda\,(P_\Gamma^2 + \mathrm{m}\Delta T)$ where T may be choosen invariant. Then M_λ^1 can be extended by $M_\lambda^1 + \lambda^3\left(\dfrac{1}{3!}\,S_\Gamma^3 + P\Delta T\right)$ and the argument developped for $k > 1$ can be applied without changes.

If $k = 0$, m extends first by $\mathrm{m} + \lambda P + \dfrac{1}{2}\,\lambda^2 P_\Gamma^2$.

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Manuscript received: November 3, 1984.